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A NOTE ON ASYMPTOTIC PRIME SEQUENCES[1]

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ABSTRACT. The lengths of all maximal asymptotic prime sequences over an ideal in a local ring are shown to be the same. This number can be calculated in terms of analytic spread and depths of minimal primes in the completion.

Introduction. Let R be a Noetherian ring. For an ideal $I \subseteq R$, let $\bar{I}$ denote its integral closure. In [9] Rees gives the following definitions: An element x is said to be asymptotically prime to I if $(\overline{I^n}: x) = \overline{I^n}$ for all $n \geq 1$. Elements $x_1, \ldots, x_k$ are said to form an asymptotic prime sequence over I if for each $1 \leq j < k$, x_{j+1} is asymptotically prime to $(I, x_1, \ldots, x_j)$. Rees proves that if $x_1, \ldots, x_k$ is an asymptotic prime sequence over I in a local ring R, then the analytic spread of I, denoted $a(I)$, satisfies $a(I) \leq \dim R - k$, with equality holding when R is quasiunmixed and the given sequence is maximal. The purpose of this note is to show that for any local ring R and ideal $I \subseteq R$, the lengths of all maximal asymptotic prime sequences over I are the same and equals the number

$$s(I) = \min\{\dim R^*/z^* - a(IR^* + z^*/z^*) \mid z^* \text{ is a minimal prime ideal in } R^*\text{—the completion of } R\}.$$

We accomplish this by showing that the property of being an asymptotic prime sequence is preserved upon passing to the completion and moding out by a minimal prime. In this context the ring is quasiunmixed and such sequences are easier to handle.

1.0. The following terminology will be used throughout with little or no subsequent reference.

1.1 *Terminology.* R will always denote a local Noetherian ring with maximal ideal M and completion R^*. R is quasiunmixed in case $\dim R^*/z^* = \dim R$ for each minimal prime $z^* \subseteq R^*$. For an ideal $I \subseteq R$, $\bar{I}$, the integral closure of I, is the set $\{x \in R \mid x^n + i_1 x^{n-1} + \cdots + i_n = 0 \text{ for some } n, \text{ with } i_r \in I^r, 1 \leq r \leq n\}$. It is well known that $\bar{I}$ is an ideal of R. $\bar{A}^*(I)$ will denote the set of prime ideals $\{P \subseteq R \mid P \in \operatorname{Ass}(R/\overline{I^k}) \text{ for some } k\}$. Thus, elements $x_1, \ldots, x_k$ form an asymptotic prime sequence over I iff for each j, $x_{j+1} \notin \bigcup\{P \mid P \in \bar{A}^*(I, x_1, \ldots, x_j)\}$. For an ideal $I \subseteq R$, $R(I)$ will denote $R[It, t^{-1}]$, the Rees ring of R wrt I—a graded subring of

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$R[t, t^{-1}]$, t an indeterminate. Denoting t^{-1} by u, then it is known [6] that $P \in \bar{A}^*(I)$ iff there exists $Q \in \bar{A}^*(uR(I))$ such that $Q \cap R = P$. Furthermore, a prime Q in $R(I)$ is minimal iff there exists a minimal prime $P \subseteq R$ such that $Q = PR[t, t^{-1}] \cap R(I)$. In this case $R(I)/Q \cong R(I + P/P)$, the Rees ring of R/P wrt $I + P/P$. Finally, $a(I)$ will denote the analytic spread of I—the maximal number of elements analytically independent in I (see [5]).

1.2 Remark. Lemmas 1.4 and 1.5 below were given independently by the author in [3] and Professor L. J. Ratliff Jr. in [8]. In [8] Professor Ratliff proves several theorems about asymptotic prime sequences which are analogous to theorems about R-sequences. Lemma 1.6 is extracted from Theorem 2.6 in [9], but is proven directly here for the sake of exposition.

1.3 Lemma. *Let $I \subseteq R$ be an ideal. Let $z_1, \ldots, z_r$ be the minimal primes of R. Then $\bigcup_{i=1}^{r} z_i \subseteq \bigcup\{P \mid P \in \bar{A}^*(I)\}$.*

Proof. Let z_i be a minimal prime and choose $x \in z_i$. Then there exists $y \in \bigcap_{j \neq i} z_j - z_i$ such that $x \cdot y \in Z =$ nilradical of R. Theorem 7 in [2] implies that $\bigcap_{n \geq 1} \overline{I^n} = Z$, so that if $x \notin \bigcup\{P \mid P \in \bar{A}^*(I)\}$ then $y \in \bigcap_{n \geq 1} \overline{I^n} = Z \subseteq z_i$. But this is a contradiction.

1.4 Lemma. *Let $I \subseteq R$ be an ideal. For a prime $P \subseteq R$, $P \in \bar{A}^*(I)$ iff there exists a prime $P^* \subseteq R^*$ such that $P^* \in \bar{A}^*(IR^*)$ and $P^* \cap R = P$.*

Proof. See [3 or 8].

1.5 Lemma. *Let $I \subseteq R$ be an ideal. Given a prime $P \subseteq R$, then $P \in \bar{A}^*(I)$ iff there exists a minimal prime $z \subseteq R$ such that $P/z \in \bar{A}^*(I + z/z)$ in R/z.*

Proof. One direction follows easily from the fact that an element $x \in \overline{I^k}$ iff for each minimal prime $z \subseteq R$, the image of x in R/z is in $\overline{(I^k + z/z)}$. For the other direction, suppose P, z are primes of R such that z is minimal and $P/z \in \bar{A}^*(I + z/z)$. By 1.4 we may assume that R is complete. Let $R(I)$ be the Rees ring of R wrt I and let $R(I/z)$ be the Rees ring of R/z wrt $I + z/z$. As noted in 1.1, $z_1 = zR[t, t^{-1}] \cap R(I)$ is a minimal prime of $R(I)$ such that $R(I)/z_1 \cong R(I/z)$. Moreover, there exists Q in $R(I/z)$ such that $Q \in \bar{A}^*(u \cdot R(I/z))$ and $Q \cap R/z = P/z$. Since R/z is quasiunmixed, 2.3 in [7] implies that Q is minimal over $u \cdot R(I/z)$. Under the above, $\cong$ Q corresponds to a prime $q \subseteq R(I)$ minimal over $u \cdot R(I) + z_1$, which by [3 or 8] implies that $q \in \bar{A}^*(u \cdot R(I))$. Thus $q \cap R = P \in \bar{A}^*(I)$.

1.6 Lemma (cf. 2.6 in [9]). *Let $I \subseteq R$ be an ideal. Let $x \notin \bigcup\{P \mid P \in \bar{A}^*(I)\}$. Then $a(I, x) = a(I) + 1$.*

Proof. Let Y be an indeterminate. It is shown in [3] that for a prime $P \subseteq R$, $P \in \bar{A}^*(I)$ iff $PR[Y] \in \bar{A}^*(IR[Y])$. Since $a(I) = a(IR[Y])$, we may therefore pass to the ring $R[Y]_{MR[Y]}$ and assume that R/M is infinite. Under this assumption Northcott and Rees prove in [5] that $a(I) =$ minimal number of generators of a minimal reduction of I. Since their work shows that reductions of I have the same integral closures as I, we may assume that I is generated by $a(I)$ analytically independent elements. Now, let $I = (x_1, \ldots, x_k)$ and choose $x \notin \bigcup\{P \mid P \in \bar{A}^*(I)\}$. Suppose f is a form of degree d with $f(x_1, \ldots, x_k, x) = 0$. We must show f has all its coefficients in M. Write $f = r_1 N_1 + \cdots + r_n N_n$ as an R-linear combination of monomials N_i. Let Q be the ideal $(M, u)R(I)$ in $R(I)$. Since the x_j are analytically independent, Q is a prime ideal and we may localize $R(I)$ at Q. In $R(I)_Q$ we may write $x_i = (x_i \cdot t)u$, so

$$\begin{aligned} 0 = f(x_1, \ldots, x_k, x) &= f(x_1 t \cdot u, \ldots, x_k t \cdot u, x) \\ &= r_1' u^{d-e_1} x^{e_1} + \cdots + r_n' u^{d-e_n} x^{e_n}, \end{aligned}$$

where for each j, r_j' is the element $r_j N_j'(x_1, \ldots, x_k) t^{d-e_j}$ of $R(I)_Q$, with $N_j'(x_1, \ldots, x_k)$ such that $N_j'(x_1, \ldots, x_k) x^{e_j} = N_j(x_1, \ldots, x_k, x)$. Now 1.1 and the choice of x imply that $x \notin \bigcup\{P \mid P \in \bar{A}^*(u \cdot R(I)_Q)\}$. Furthermore, since primes minimal over $u \cdot R(I)_Q$ belong to $\bar{A}^*(u \cdot R(I)_Q)$, we have that u, x form part of a system of parameters for $R(I)_Q$, and are therefore analytically independent. Thus $r_k' \in Q$ for all j. Collecting terms which correspond to t^{d-e_j} we have a sum of monomials in $x_1, \ldots, x_k$ belonging to MI^{d-e_j}. By the analytic independence of the x_i in R, each coefficient lies in M.

1.7 DEFINITION. Let $I \subseteq R$ be an ideal. If $x_1, \ldots, x_k$ form an asymptotic prime sequence over I, we say that $x_1, \ldots, x_k$ form a maximal asymptotic prime sequence over I if $M \in \bar{A}^*(I, x_1, \ldots, x_k)$.

1.8 LEMMA. *Let $I \subseteq R$ be an ideal. Let $x_1, \ldots, x_k$ form an asymptotic prime sequence over I. Then for every minimal prime $z^* \subseteq R^*$, the images of $x_1, \ldots, x_k$ in R^*/z^* form an asymptotic prime sequence over $IR^* + z^*/z^*$. Moreover, if for some z^*, the images of $x_1, \ldots, x_k$ in R^*/z^* form a maximal asymptotic prime sequence over $IR^* + z^*/z^*$, then $x_1, \ldots, x_k$ form a maximal asymptotic prime sequence over I.*

PROOF. By 1.3 the images of $x_1, \ldots, x_k$ survive in each R^*/z^*. The result is now clear by 1.4 and 1.5.

1.9 THEOREM. *Let $I \subseteq R$ be an ideal. Then the lengths of all maximal asymptotic prime sequences over I are the same and equals the number*

$$s(I) = \min\{\dim R^*/z^* - a(IR^* + z^*/z^*) \mid z^* \textit{ is a minimal prime in } R^*\}.$$

PROOF. Let $x_1, \ldots, x_k$ be a maximal asymptotic prime sequence over I such that k is the least number of elements in such sequences. Let $y_1, \ldots, y_s$ be any other maximal asymptotic prime sequence over I. We will show $k = s$. By definition, $M \in \bar{A}^*(I, x_1, \ldots, x_k)$. 1.4 implies that $M^* \in \bar{A}^*((I, x_1, \ldots, x_k)R^*)$. 1.5 implies that there is a minimal prime $z^* \subseteq R^*$, such that if we write T for R^*/z^*, then $MT \in \bar{A}^*((I, x_1, \ldots, x_k)T)$. As T is quasiunmixed (and therefore satisfies the altitude formula), Theorem 3 in [4] implies that $a((I, x_1, \ldots, x_k)T) = \dim T$. Since the images of $x_1, \ldots, x_k$ in T form an asymptotic prime sequence over IT (by 1.8), 1.6 implies that $a(IT) = \dim T - k$. Beginning with y_1, 1.6 applied k times in T shows that $a((I, y_1, \ldots, y_k)T) = \dim T$. Again Theorem 3 in [4] implies that $MT \in \bar{A}^*((I, y_1, \ldots, y_k)T)$ so by 1.8, $k = s$. Finally the argument given indicates clearly that $k \geqslant s(I)$, and may be repeated to provide a contradiction if $k > s(I)$.

1.10 COROLLARY (CF. [9]). *If $x_1, \ldots, x_k$ form an asymptotic prime sequence over I, then $a(I) \leq \dim R - k$ and equality holds when R is quasiunmixed and the sequence is maximal.*

PROOF. Immediate from 1.9 and the following facts:

(i) $a(I) = a(IR^*)$.

(ii) There exists a minimal prime $z^* \subseteq R^*$ such that $a(IR^*) = a(IR^* + z^*/z^*)$ (see [**1**]).

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